

Stratonovich's Signatures of Brownian Motion Determine Brownian Sample Paths

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Abstract. The signature of Brownian motion in \mathbb{R}^d over a running time interval $[0, T]$ is the collection of all iterated Stratonovich path integrals along the Brownian motion. We show that, in dimension $d \geq 2$, almost all Brownian motion sample paths (running up to time T) are determined by its signature over $[0, T]$.¹

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1 Introduction

Let $W = (W_t^1, \dots, W_t^d)_{t \geq 0}$ be a Brownian motion in the Euclidean space of dimension $d \geq 2$. The Stratonovich signature of W over the duration from time 0 to time T , according to K. T. Chen [5] and T. Lyons [6], is the formal series with d indeterminates X_1, \dots, X_d whose coefficients are iterated Stratonovich's path integrals of Brownian sample paths:

$$S(W)_{[0,T]} = \sum_{n=0}^{\infty} \sum_{\pi \in S_n} [\pi_1 \cdots \pi_n]_{0,T} X_{\pi_1} \cdots X_{\pi_n} \quad (1.1)$$

where S_n denotes the permutation group of $\{1, \dots, n\}$, $\sum_{\pi \in S_n}$ runs through permutations $\pi = (\pi_1, \dots, \pi_n) \in S_n$, and the square bracket $[\pi_1 \cdots \pi_n]_{s,t}$ denotes the multiple Stratonovich integral of Brownian motion over $[s, t]$, i.e.

$$[\pi_1 \cdots \pi_n]_{s,t} = \int_{s < t_1 < \cdots < t_n < t} \circ dW_{t_1}^{\pi_1} \circ \cdots \circ dW_{t_n}^{\pi_n}. \quad (1.2)$$

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These integrals may be defined by means of Itô's integration. In fact, multiple integrals may be defined inductively by

$$[\pi_1 \cdots \pi_n]_{s,t} = \int_s^t [\pi_1 \cdots \pi_{n-1}]_{[s,r]} \circ dW_r^{\pi_n}$$

where $\circ d$ indicates the integration in Stratonovich's sense, which in turn can be expressed in terms of Itô's and ordinary integrals.

If one is not concerned about underlying algebraic structures defined by iterated integrals, it is not necessary to approach the Stratonovich signature through the formal series (1.1). We consider the collection of all possible iterated Stratonovich integrals $[\pi_1 \cdots \pi_n]_{0,T}$, emphasizing the fact that they are all taken over a fixed time interval $[0, T]$, as the Stratonovich signature(s) of Brownian motion (over $[0, T]$). Since we will work on signatures over a fixed interval, the lower script 0 and T will be omitted if no confusion may arise, for the sake of simplicity of notations. Without losing generality we may from now on assume that $T = 1$.

Since the notion of signatures is so significant in this paper, we thus would like to present a formal definition.

Definition 1.1 *Let $W = (W_t)_{t \geq 0}$ be a Brownian motion in \mathbb{R}^d starting at 0. Then the Stratonovich signature (or signatures) of W over $[0, 1]$ is the collection of all iterated Stratonovich integrals*

$$[j_1 \cdots j_n] = \int_{0 < t_1 < \cdots < t_n < 1} \circ dW_{t_1}^{j_1} \circ \cdots \circ dW_{t_n}^{j_n}$$

where n runs through $1, 2, \dots$, and $j_1, \dots, j_n \in \{1, \dots, d\}$.

The interest for signatures of paths has a long history. First of all, sequences of multiple iterated integrals arise naturally in Picard's iteration of solving ordinary differential equations. Multiple iterated integrals of Brownian motion appeared already in early 1930's in Wiener's celebrated work on harmonic analysis on the Wiener space, and K. Itô studied them in terms of his integration theory. Meanwhile, from 1950's to late 1970's, in a series of articles [4], [5], [2], [3] etc. K. T. Chen demonstrated the usefulness of iterated integrals along piecewise smooth paths in manifolds. K. T. Chen showed the interesting algebraic structures defined by sequences of iterated integrals, developed a representation theory, and established a homotopy theory in terms

of iterated integrals. The importance of multiple Stratonovich integrals, however, surprisingly was not recognized until the important contributions by Wong-Zakai [12], Ikeda and Watanabe [8], in which the convergence theorem for solutions to stochastic differential equations in Stratonovich's sense was proved. The definite role played by iterated Stratonovich's integrals was finally revealed in T. Lyons [10] (also see [9]) in which a universal limit theorem for solutions of Stratonovich's stochastic differential equations was proved. T. Lyons has realized that the key elements for defining an integration theory along a continuous path which is not necessary piecewise smooth is the sequence of iterated integrals that must be specified. This idea led to the discovery of the p -variation metric among continuous paths with finite variations, which allows to develop the theory of rough paths.

It has been conjectured that the signature of a path over a fixed time duration $[0, 1]$, which can be read out at the terminal time 1, should be a good summary of information about the flow of timely ordered events, recorded in its path during time 0 to time 1. K. T. Chen [3] first proved that indeed it is possible to recover the whole path (up to tree-like components of the path which are not counted in its signature) by reading its signature. B. Hambly and T. Lyons [6] extended and quantified Chen's result to rectifiable curves in multi-dimensional spaces. Unfortunately, these results are not applicable to interesting random curves, since, for example, almost all sample paths of a non-trivial diffusion process are not rectifiable.

In this article, we demonstrate that for $d \geq 2$ almost all d -dimensional Brownian paths can be recovered from its Stratonovich's signature. In other words, theoretically, all information recorded in Brownian motion from 0 to 1 can be read out from the Stratonovich signature over $[0, 1]$.

To state our main result more precisely, we need to introduce more notations. Let $\mathcal{F}_t^0 = \sigma\{W_s : s \leq t\}$ be the filtration generated by W , and \mathcal{F}_1 be the completion of \mathcal{F}_1^0 (under the Brownian measure P), and \mathcal{G}_1 be the complete σ -algebra generated by the Stratonovich signatures, i.e. the completion of the σ -algebra $\sigma\{[\pi_1 \cdots \pi_n]_{0,1} : \pi \in S_n; n \in \mathbb{N}\}$.

Our main result may be stated as follows

Theorem 1.2 $\mathcal{F}_1 = \mathcal{G}_1$. *Therefore the Stratonovich signature determines Brownian sample paths almost surely.*

To prove this theorem, we need to develop a method of reconstructing almost all Brownian sample paths given their signatures. We will come to this point shortly.

In order to appreciate why Stratonovich signatures are able to represent the sample paths of Brownian motion, let us look at how to obtain iterated integrals of smooth differential forms along Brownian motion paths in terms of the Stratonovich signatures. The remarkable fact, which certainly goes back to K. T. Chen [3] for the deterministic case, is that any polynomials of Brownian motion (evaluated at a fixed time 1) is a linear combination of the signatures over $[0, 1]$. In fact

$$W_t^{j_1} \cdots W_t^{j_n} = \sum_{\pi \in S_n} [j_{\pi_1} \cdots j_{\pi_n}]_{0,t}. \quad (1.3)$$

This formula can be proved by integrating by parts:

$$W_t^{j_1} W_t^{j_2} = [j_1 j_2]_{0,t} + [j_2 j_1]_{0,t}$$

and for $n \geq 2$

$$\begin{aligned} W_t^{j_1} \cdots W_t^{j_n} W_t^{j_{n+1}} &= \sum_{\pi \in S_n} \int_0^t [j_{\pi_1} \cdots j_{\pi_n}]_{0,s} \circ dW_s^{j_{n+1}} \\ &\quad + \sum_{\pi \in S_n} \int_0^t W_s^{j_{n+1}} \circ d[j_{\pi_1} \cdots j_{\pi_n}]_{0,s} \\ &= \sum_{\pi \in S_n} [j_{\pi_1} \cdots j_{\pi_n} j_{n+1}]_{0,t} \\ &\quad + \sum_{\pi \in S_n} \int_0^t W_s^{j_{n+1}} [j_{\pi_1} \cdots j_{\pi_{n-1}}]_s \circ dW_s^{j_{\pi_n}} \end{aligned}$$

and (1.3) follows. If $\alpha^1, \dots, \alpha^k$ are smooth differential forms on \mathbb{R}^d with compact supports, then iterated Stratonovich integrals $[\alpha^1 \cdots \alpha^k]_{s,t}$ are defined inductively by

$$[\alpha^1 \cdots \alpha^k]_{s,t} = \int_s^t [\alpha^1 \cdots \alpha^{k-1}]_{s,u} \alpha^k(\circ dW_u).$$

Since polynomials are dense in C^k functions for any k under uniform convergence over compact subsets, therefore all iterated Stratonovich integrals of 1-forms against W are measurable functionals of the signatures. This is the context of the following lemma.

Lemma 1.3 *If $\alpha^1, \dots, \alpha^k$ are smooth differential forms on \mathbb{R}^d with compact supports, then $[\alpha^1 \cdots \alpha^k]_{0,1}$ is \mathcal{G}_1 -measurable.*

Proof. If α^l have polynomial coefficients, then we have seen that $[\alpha^1 \cdots \alpha^k]_{0,1}$ is a linear combination of the Stratonovich signatures, so it is \mathcal{G}_1 -measurable. In general case, we may approximate $\alpha^1, \dots, \alpha^k$ by polynomials $\alpha_n^1, \dots, \alpha_n^k$ in C^{k+1} norm, so that

$$[\alpha_n^1 \cdots \alpha_n^k]_{s,t} \rightarrow [\alpha^1 \cdots \alpha^k]_{s,t}$$

in $L^2(\Omega, \mathcal{F}, P)$. This yields that $[\alpha^1 \cdots \alpha^k]_{0,T}$ is \mathcal{G}_T -measurable. ■

These iterated Stratonovich integrals $[\alpha^1 \cdots \alpha^k]_{0,1}$ may be considered as "extended" signatures of W over $[0, 1]$.

Since there is no essential differences in our proof of Theorem 1.2 between dimension two and the higher dimensional case, we therefore concentrate on the case $d = 2$. The main idea and the key steps in the proof of Theorem 1.2 are described as follows.

To construct approximations of Brownian motion W in terms of a countable family of extended signatures, for each $\varepsilon > 0$ we construct an ε -grid so that \mathbb{R}^2 is divided into squares with center at $\mathbf{z}\varepsilon = (z_1\varepsilon, z_2\varepsilon)$ and wide ε , and let

$$S_{\mathbf{z}} = \{(x_1, x_2) : |x_1 - z_1\varepsilon| + |x_2 - z_2\varepsilon| \leq \frac{1}{2}\varepsilon(1 - \varepsilon)\}$$

which is strictly located inside the squares with the same center. We naturally construct an approximation by polygons which join the centers of the squares $S_{\mathbf{z}}$ which have been visited by the Brownian motion paths W . It is not very difficult to show these polygons converge to Brownian motion paths almost surely, and we want to show that these polygonal approximations are indeed determined by the Stratonovich signatures of W . To this end, we construct a smooth differential 1-form $\phi^{\mathbf{z}}$ which has a compact support inside the squares $S_{\mathbf{z}}$ so that for different indices $\mathbf{z} \in \mathbb{Z}^2$, these differential 1-forms $\phi^{\mathbf{z}}$ have disjoint supports. The key observation is that the Stratonovich integral $\int \phi^{\mathbf{z}}(\circ dW)$ does not vanish almost surely over the duration that the Brownian motion has visited $S_{\mathbf{z}}$. This crucial fact allows us to identify those squares the Brownian motion has visited entirely in terms of the signatures of the Brownian motion.

2 Several technical facts

In this section we establish several technical facts which will be used in the proof of Theorem 1.2.

A planer square is a nice domain but its boundary has four corners and thus is not C^1 . For the technical reasons we consider a domain obtained from a square by replacing the portion of the boundary near each corner by a quarter of small circles. More precisely, for a small $\frac{1}{4} > \varepsilon > 0$, and, as we will use this parameter ε for other constructions, for $\beta \gg 1$, let

$$D = \left\{ (x_1, x_2) : 0 \leq x_1, x_2 \leq \frac{1}{2} \right\} \setminus \left\{ \left| x_1 - \frac{1}{2} + \varepsilon^\beta \right|^2 + \left| x_2 - \frac{1}{2} + \varepsilon^\beta \right|^2 \geq \varepsilon^{2\beta} \right\}$$

and the typical planer domain we will handle is

$$G = \{(x_1, x_2) : (|x_1|, |x_2|) \in D\}. \quad (2.1)$$

For $a > 0$, G_a denotes the similar planer domain aG , i.e. $G_a = \{x = (x_1, x_2) : (ax_1, ax_2) \in G\}$.

Let $W_t = (W_t^1, W_t^2)$ be a two dimensional Brownian motion on a probability space (Ω, \mathcal{F}, P) , and let $a_3 > a_2 > a_1$. Let

$$S_0 = \inf\{t > 0 : W_t \in \partial G_{a_3}\},$$

$$S_1 = \inf\{t > S_0 : W_t \in \partial G_{a_1}\}$$

and

$$S_2 = \inf\{t > S_1 : W_t \in \partial G_{a_2}\}$$

which are stopping times, finite almost surely. We are interested in the distribution of the random variable $X = \int_{S_0}^\tau \phi(\circ dW_s)$, where ϕ is a differential 1-form which coincides with $x^2 dx^1$ on G_{a_2} , conditional to $\{S_1 < \tau\}$.

To this end, consider the diffusion process $X = (X^1, X^2, X^3)$ in \mathbb{R}^3 associated with the following stochastic differential equations

$$dX_t^1 = dW_t^1; \quad dX_t^2 = dW_t^2; \quad dX_t^3 = X_t^2 \circ dW_t^1. \quad (2.2)$$

It is an easy exercise to calculate the infinitesimal generator of X , which is $L = \frac{1}{2}(A_1^2 + A_2^2)$, where $A_1 = \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_3}$ and $A_2 = \frac{\partial}{\partial x_2}$. In particular, the Lie bracket $[A_1, A_2] = -\frac{\partial}{\partial x^3}$, so that L is hypoelliptic (Theorem 1.1, page 149, Hörmander [7]).

Lemma 2.1 *Let W be Brownian motion in \mathbb{R}^2 on (Ω, \mathcal{F}, P) started from a point at ∂G_{a_1} , $S = \inf\{t > 0 : W_t \in \partial G_{a_2}\}$, and $\xi = \int_0^S W_s^2 \circ dW_s^1$. Then, for any $y \in \partial G_{a_2}$, the conditional distribution $P\{\xi \in dz | W_S = y\}$ has a continuous density function in z .*

Proof. Let $D = G_{a_2} \times \mathbb{R}^1$, and $S = \inf\{t \geq 0 : X_t \notin D\}$ the first exit time of the diffusion process X . Then, D has a C^1 -boundary (this is the reason for which we use rounded squares) and the condition required in [1] is satisfied, as the normal to the boundary belongs to the plane spanned by A_1 and A_2 . Thus, according to a theorem of Ben Arous, Kusuoka and Stroock (Theorem 1.22, page 181, in [1]), the Poisson measure of L on the open domain D has a (smooth) density, which implies that the distribution of X_S has a continuous density function on ∂D with respect to the Lebesgue measure on ∂D . Therefore the conditional distribution $P\{\xi \in dz | W_S = y\}$ has a continuous density on \mathbb{R}^1 for $y \in \partial G_{a_2}$. ■

Let $f(x_1, x_2)$ be a smooth function on \mathbb{R}^2 with a support in G_{a_3} such that $f(x_1, x_2) = x_2$ on G_{a_2} . Consider the smooth differential 1-form $\phi = f(x_1, x_2)dx_1$ on \mathbb{R}^2 .

Lemma 2.2 *Under above assumptions and notations. Let $Z = \int_{S_1}^{S_2} \phi(\circ W_s)$. Then the conditional distribution of Z given $W_{S_1} = (x_1, x_2)$ and $W_{S_2} = (y_1, y_2)$ has a continuous density function, i.e.*

$$P\{Z \in dz | W_{S_1} = (x_1, x_2), W_{S_2} = (y_1, y_2)\} = p((x_1, x_2), (y_1, y_2), z)dz \quad (2.3)$$

for some nonnegative function p .

Proof. This follows from the Strong Markov property of X and the previous Lemma. ■

Lemma 2.3 *Under conditions and notations described above. Let U be an open subset such that $\overline{G_{a_3}} \cap U = \emptyset$ and $\tau = \inf\{t > S_0 : W_t \in \partial U\}$ be a hitting time. Let $T = S_2 + \tau \circ S_2$. Then the random variable $\eta = \int_{S_0}^T \phi(\circ dW_s) \neq 0$ almost surely on $\{S_1 < T\}$.*

Proof. Write

$$\eta = \int_{S_1}^{S_2} \phi(\circ dW_s) + \int_{S_0}^{S_1} \phi(\circ dW_s) + \int_{S_2}^T \phi(\circ dW_s).$$

For any stopping time S we have two σ -fields, namely \mathcal{F}_S which is the σ -algebra of events happening before S , and $\mathcal{F}_{>S}$ the σ -algebra of events depending on the path after stopping time S . By definition, $1_{\{S_1 < T\}} \int_{S_0}^{S_1} \phi(\circ dW_s)$ is \mathcal{F}_{S_1} -measurable and $1_{\{S_1 < T\}} \int_{S_2}^T \phi(\circ dW_s)$ is $\mathcal{F}_{S_1} \vee \mathcal{F}_{>S_2}$ measurable. Let $Y = \int_{S_0}^{S_1} \phi(\circ dW_s) + \int_{S_2}^T \phi(\circ dW_s)$ for simplicity. By the strong Markov property

$$\begin{aligned} E \left\{ 1_{\{S_1 < T\}} \int_{S_1}^{S_2} \phi(\circ dW_s) | \mathcal{F}_{S_1} \vee \mathcal{F}_{>S_2} \right\} &= 1_{\{S_1 < T\}} E \left\{ \int_{S_1}^{S_2} \phi(\circ dW_s) | \mathcal{F}_{S_1} \vee \mathcal{F}_{>S_2} \right\} \\ &= 1_{\{S_1 < T\}} E \left\{ \int_{S_1}^{S_2} \phi(\circ dW_s) | W_{S_1}, W_{S_2} \right\} \end{aligned}$$

so that

$$E \left\{ F(\eta) 1_{\{S_1 < T\}} | \mathcal{F}_{S_1} \vee \mathcal{F}_{>S_2} \right\} = 1_{\{S_1 < T\}} E \left\{ F \left(\int_{S_1}^{S_2} \phi(\circ dW_s) + Y \right) | \mathcal{F}_{S_1} \vee \mathcal{F}_{>S_2} \right\}.$$

Suppose $F(z + y) = \sum_j H_j(z) K_j(y)$, then

$$\begin{aligned} E \left\{ F(\eta) | \mathcal{F}_{S_1} \vee \mathcal{F}_{>S_2} \right\} &= \sum_j K_j(Y) E \left\{ H_j \left(\int_{S_1}^{S_2} \phi(\circ dW_s) \right) | \mathcal{F}_{S_1} \vee \mathcal{F}_{>S_2} \right\} \\ &= \sum_j K_j(Y) E \left\{ H_j \left(\int_{S_1}^{S_2} \phi(\circ dW_s) \right) | W_{S_1}, W_{S_2} \right\} \end{aligned}$$

and therefore

$$E \left\{ F(\eta) 1_{\{S_1 < T\}} \right\} = \sum_j E \left\{ K_j(Y) E \left[H_j \left(\int_{S_1}^{S_2} \phi(\circ dW_s) \right) | W_{S_1}, W_{S_2} \right] 1_{\{S_1 < T\}} \right\}.$$

Since $\int_{S_1}^{S_2} \phi(\circ dW_s)$ has a conditional probability density $p(x, y, z)$

$$E \left[1_{\{S_1 < T\}} \int_{S_1}^{S_2} \phi(\circ dW_s) \in dz | W_{S_1} = x, W_{S_2} = y \right] = p(x, y, z) dz$$

and thus

$$\begin{aligned} E \left\{ F(\eta) 1_{\{S_1 < T\}} \right\} &= E \left\{ 1_{\{S_1 < T\}} \int_{\mathbb{R}} \sum_j K_j(Y) H_j(z) p(W_{S_1}, W_{S_2}, z) dz \right\} \\ &= E \left\{ 1_{\{S_1 < T\}} \int_{\mathbb{R}} F(Y + z) p(W_{S_1}, W_{S_2}, z) dz \right\}. \end{aligned}$$

In particular $P\{\eta = 0, S_1 < T\} = 0$. ■

3 Constructing approximations to Brownian paths

In this section, we construct polygonal approximations to the planer Brownian motion sample paths by tracing the sample paths of Brownian motion through prescribed ε -grids laid out in the plane. Our construction equally applies to higher dimensional Brownian motion with only minor modifications which we will leave to the reader.

To make our arguments clear, let us work with the classical Wiener space $(\mathbf{W}, \mathcal{B}, P)$, where \mathbf{W} is the space of all continuous paths in \mathbb{R}^2 started at 0, \mathcal{B} is the Borel σ -algebra on \mathbf{W} and P is the unique probability so that the coordinate process $W = (W^1, W^2)$ is a planer Brownian motion on $(\mathbf{W}, \mathcal{B}, P)$ started at 0.

Let $\varepsilon \in (0, \frac{1}{4})$. Recall that G is the planer domain defined by (2.1) which is the planer square with corners rounded. For $\mathbf{z} = (z_1, z_2) \in \mathbb{Z}^2$ we assign three boxes $H_{\mathbf{z}}^\varepsilon \subset K_{\mathbf{z}}^\varepsilon \subset Z_{\mathbf{z}}^\varepsilon$ which are all similar domains to G , with a common center $\varepsilon\mathbf{z}$ lies on the $\varepsilon\mathbb{Z}$:

$$\begin{aligned} H_{\mathbf{z}}^\varepsilon &= \varepsilon\mathbf{z} + \varepsilon(1 - \varepsilon)G, \\ K_{\mathbf{z}}^\varepsilon &= \varepsilon\mathbf{z} + \varepsilon \left(1 - \varepsilon + \frac{\varepsilon\varphi(\varepsilon)}{2} \right) G, \\ Z_{\mathbf{z}}^\varepsilon &= \varepsilon\mathbf{z} + \varepsilon(1 - \varepsilon + \varepsilon\varphi(\varepsilon)) G, \end{aligned}$$

and

$$V_{\mathbf{z}}^\varepsilon = \varepsilon\mathbf{z} + \varepsilon G$$

where $\varphi(\varepsilon) \ll \varepsilon^\alpha$ (with $\alpha \geq 10$) but to be chosen late on.

Let us notice that the gap between $Z_{\mathbf{z}}^\varepsilon$ and the box $V_{\mathbf{z}}^\varepsilon$ has a magnitude $\varepsilon^2(1 - \varphi(\varepsilon))$, while the magnitude of the gap between $H_{\mathbf{z}}^\varepsilon$ and $K_{\mathbf{z}}^\varepsilon$ is $\frac{1}{2}\varepsilon^2\varphi(\varepsilon)$. Since $\varphi(\varepsilon) \ll \varepsilon^\alpha$ so that

$$\varepsilon^2(1 - \varphi(\varepsilon)) \gg \frac{1}{2}\varepsilon^2\varphi(\varepsilon)$$

a crucial fact we will use below.

If $A \subset \mathbb{R}^2$, then T_A denotes the hitting time of A by the Brownian motion W .

Lemma 3.1 *There is $\varphi(\varepsilon) \ll \varepsilon^\alpha$ (with $\alpha \geq 11$) and $\beta \gg 10$ such that for every $\mathbf{z} = (z_1, z_2) \in \mathbb{Z}^2$ and $x \in \partial Z_{\mathbf{z}}^\varepsilon$*

$$P\{T_{\partial V_{\mathbf{z}}^\varepsilon} < T_{H_{\mathbf{z}}^\varepsilon} | W_0 = x\} \leq \varepsilon^{10}. \quad (3.1)$$

Proof. We need to show that the probability on the left-hand side is dominated by the ratio of the distances between x to $\partial H_{\mathbf{z}}^\varepsilon$ and to $\partial V_{\mathbf{z}}^\varepsilon$ which is $\frac{\varphi(\varepsilon)}{1-\varphi(\varepsilon)}$, which in turn yields the bound in (3.1) as $\varepsilon < \frac{1}{4}$ by increasing α to kill any possible constant appearing in the domination. This is standard for one dimensional Brownian motion. Similar estimates may be obtained by means of potential theory. Clearly the left-hand side of (3.1) does not depend on $\mathbf{z} \in \mathbb{Z}^2$ so let us assume $\mathbf{z} = 0$. Let u be the unique harmonic function on $V_{\mathbf{z}}^\varepsilon \setminus H_{\mathbf{z}}^\varepsilon$ such that $u = 1$ on $\partial V_{\mathbf{z}}^\varepsilon$ and $u = 0$ on $\partial H_{\mathbf{z}}^\varepsilon$. Then, $u(W_{t \wedge T_{\partial V_{\mathbf{z}}^\varepsilon} \wedge T_{H_{\mathbf{z}}^\varepsilon}})$ is a bounded martingale, so that

$$u(x) = P\{T_{\partial V_{\mathbf{z}}^\varepsilon} < T_{H_{\mathbf{z}}^\varepsilon} | W_0 = x\}.$$

By the uniform continuity of the potential u with respect to the distance of x to the interior boundary $\partial H_{\mathbf{z}}^\varepsilon$ (for example see sections 4-2 in Port and Stone [11]), we may chose $\varphi(\varepsilon)$ small enough so that x is closer to $\partial H_{\mathbf{z}}^\varepsilon$ than to $\partial V_{\mathbf{z}}^\varepsilon$, to ensure that $u(x) \leq \varepsilon^{10}$ as long as $x \in \partial Z_{\mathbf{z}}^\varepsilon$.

To see the magnitude, we can consider the harmonic function on the disk centered at 0 with radius ε

$$w(x_1, x_2) = \frac{1}{\log(1 + 2\varepsilon - \varepsilon^2)} \log \left(\frac{x_1^2 + x_2^2}{\varepsilon^2} + 2\varepsilon - \varepsilon^2 \right)$$

which vanishes on $\rho \equiv \sqrt{x_1^2 + x_2^2} = \varepsilon(1 - \varepsilon)$ and is 1 on $\rho = \varepsilon$. At $\rho = \varepsilon(1 - \varepsilon) + \varepsilon\varphi(\varepsilon)$

$$\begin{aligned} w(x_1, x_2) &= \frac{1}{\log(1 + 2\varepsilon - \varepsilon^2)} \log(1 + 2\varphi(\varepsilon) + \varphi(\varepsilon)^2 - 2\varepsilon\varphi(\varepsilon)) \\ &\leq C \frac{\varphi(\varepsilon)}{\varepsilon}. \end{aligned}$$

Similar estimates hold for our rounded squares. In dimension 2, this can be done by a proper conformal transformation. ■

In what follows we choose such φ and β so that (3.1) holds for small $\varepsilon \in (0, 1/4)$.

For each path $w \in \mathbf{W}$, define a sequence $\{\tau_k(w) : k = 0, 1, 2, \dots\}$ of stopping times which trace the crossings of the path w through the ε -grid lattice $\varepsilon\mathbb{Z}^2$. Let $\tau_0(w) = 0$ and $\mathbf{n}_0(w) = (0, 0)$, and define $\tau_k(w)$ and $\mathbf{n}_k(w)$ inductively by

$$\tau_k(w) = \inf \left\{ t > \tau_{k-1}(w) : w_t \in \bigcup_{\mathbf{z} \neq \mathbf{n}_{k-1}(w)} H_{\mathbf{z}}^\varepsilon \right\}$$

and $\mathbf{n}_k(w) \in \mathbb{Z}^2$ such that $w(\tau_k(w)) \in H_{\mathbf{n}_k(w)}^\varepsilon$ if $\tau_k(w) < \infty$, and $\mathbf{n}_{k+1}(w) = \mathbf{n}_k(w)$ if $\tau_k(w) = \infty$. Then $\{\tau_k : k = 0, 1, \dots\}$ is a strictly increasing sequence of stopping times, and $\tau_k \uparrow \infty$ almost surely as $k \uparrow \infty$.

Let us use $\{\zeta_k : k = 0, 1, \dots\}$ and $\{\mathbf{m}_k : k = 0, 1, \dots\}$ to denote the corresponding sequences obtained in the previous definition with box $H_{\mathbf{z}}^\varepsilon$ replaced by $Z_{\mathbf{z}}^\varepsilon$. In other words

$$\zeta_k(w) = \inf \left\{ t > \zeta_{k-1}(w) : w_t \in \bigcup_{\mathbf{z} \neq \mathbf{m}_{k-1}(w)} Z_{\mathbf{z}}^\varepsilon \right\}$$

etc.

Let $M_H(w) = \inf\{k : \tau_{k+1}(w) > 1\}$ and $M_Z(w) = \inf\{k : \zeta_{k+1}(w) > 1\}$. Then both $M_H < \infty$ and $M_Z < \infty$ almost surely. Since a path which hits the box $H_{\mathbf{z}}^\varepsilon$ must first hit the larger one $Z_{\mathbf{z}}^\varepsilon$ so that $\zeta_k \leq \tau_k$ for any k , and therefore $M_H \leq M_Z$. The last inequality says a continuous path at least hit as many larger boxes than smaller ones.

Let us construct $w(\varepsilon)$ to be the polygon assuming the point $\mathbf{n}_k \varepsilon$ at time τ_k , that is,

$$w(\varepsilon)_t = \varepsilon \mathbf{n}_{k-1}(w) + \frac{t - \tau_{k-1}(w)}{\tau_k(w) - \tau_{k-1}(w)} \varepsilon \mathbf{n}_k(w) \quad \text{if } t \in [\tau_{k-1}(w), \tau_k(w)]$$

for $l = 0, 1, \dots$. We show that $w(\varepsilon)$ converges to the Brownian curves almost surely as $\varepsilon \downarrow 0$.

Lemma 3.2 *Let $W = (W_t)_{t \geq 0}$ be a planer Brownian motion started at some point inside the box $H_{\mathbf{0}}^\varepsilon$, and*

$$\tau = \inf \left\{ t > 0 : W_t \in \bigcup_{\mathbf{z} \neq \mathbf{0}} H_{\mathbf{z}}^\varepsilon \right\}.$$

Then

$$P \left\{ \sup_{0 \leq t \leq \tau} |W_t| > 3\sqrt{2}\varepsilon \right\} \leq \left(\frac{1}{3} \right)^{\left[\frac{1}{2\varepsilon} \right]}.$$

Proof. Let $\mathbf{z} = (z_1, z_2) \in \mathbb{Z}^2$ be the random variable such that $W_\tau \in H_{\mathbf{z}}^\varepsilon$.
If

$$\mathbf{z} \neq (\pm 1, \pm 1), (\pm 1, 0) \text{ or } (0, \pm 1)$$

or the Brownian motion W runs out off the square $[-3\varepsilon, 3\varepsilon] \times [-3\varepsilon, 3\varepsilon]$, then W must travel through a narrow strip of wideness ε^2 and length $\varepsilon - 2\varepsilon^\beta$, so that the probability

$$P \{ \mathbf{z} \neq (\pm 1, \pm 1), (\pm 1, 0) \text{ or } (0, \pm 1) \} \leq \left(\frac{1}{3} \right)^{\left[\frac{1}{2\varepsilon} \right]}.$$

Therefore

$$P \left\{ \sup_{0 \leq t \leq \tau} |W_t| > 3\sqrt{2}\varepsilon \right\} \leq \left(\frac{1}{3} \right)^{\left[\frac{1}{2\varepsilon} \right]}.$$

■

Lemma 3.3 *There is a sequence $\varepsilon_n \downarrow 0$, such that*

$$P \left\{ w : \lim_{n \rightarrow \infty} \inf_{\sigma} \sup_{0 \leq t \leq 1} |w_t - w(\varepsilon_n)_{\sigma(t)}| = 0 \right\} = 1$$

where \inf_{σ} takes over all possible parametrization.

Proof. We need to estimate the numbers of the crossings between different $H_{\mathbf{z}}^\varepsilon$ during the time 0 to 1. Note that

$$\begin{aligned} P \{ M_H \geq k \} &\leq P \left\{ \text{at least for one } l, \tau_{l+1} - \tau_l \leq \frac{1}{k} \right\} \\ &\leq P \left\{ \sup_{0 < t \leq \frac{1}{k}} |w_t| \geq 2\varepsilon^2 \right\} \leq \mathbb{P} \left\{ \sup_{0 < t \leq \frac{1}{k}} |w_t^1| \geq \varepsilon^2 \right\} \\ &\leq \exp \left(-\frac{\varepsilon^4}{2} k \right). \end{aligned}$$

Therefore

$$\begin{aligned}
& P \left\{ \sup_l \sup_{\tau_l \leq t \leq \tau_{l+1}} |w_t - \varepsilon \mathbf{n}_l| > 3\sqrt{2}\varepsilon \right\} \\
& \leq P \left\{ \sup_l \sup_{\tau_l \leq t \leq \tau_{l+1}} |w_t - \varepsilon \mathbf{n}_l| > 3\sqrt{2}\varepsilon : M \leq k \right\} + P \{M > k\} \\
& \leq k \left(\frac{1}{3} \right)^{\left\lfloor \frac{1}{2\varepsilon} \right\rfloor} + \exp \left(-\frac{\varepsilon^4}{2} k \right)
\end{aligned}$$

by choosing $k = \frac{1}{\varepsilon^6}$ to obtain

$$\begin{aligned}
& P \left\{ \sup_l \sup_{\tau_l \leq t \leq \tau_{l+1}} (|w_t - \varepsilon \mathbf{n}_l|) > 3\sqrt{2}\varepsilon \right\} \\
& \leq \frac{1}{\varepsilon^6} \left(\frac{1}{3} \right)^{\left\lfloor \frac{1}{2\varepsilon} \right\rfloor} + \exp \left(-\frac{1}{2\varepsilon^2} \right)
\end{aligned}$$

so by the Borel-Cantelli lemma, $w(\varepsilon_n) \rightarrow w$ almost surely for a properly chosen ε_n such that

$$\sum \frac{1}{\varepsilon_n^6} \left(\frac{1}{3} \right)^{\left\lfloor \frac{1}{2\varepsilon_n} \right\rfloor} + \exp \left(-\frac{1}{2\varepsilon_n^2} \right) < \infty.$$

■

On the other hand the gap between two boxes $H_{\mathbf{z}}^\varepsilon$ and $Z_{\mathbf{z}}^\varepsilon$ in comparison to the gap between $Z_{\mathbf{z}}^\varepsilon$ and $V_{\mathbf{z}}^\varepsilon$ is so small, it happens that $M_H = M_Z$ and $\mathbf{n}_k = \mathbf{m}_k$ on $\{k \leq M_H = M_Z\}$ with a large probability, which is the context of the following lemma.

Lemma 3.4 *For any $\varepsilon \in (0, \frac{1}{4})$ we have*

$$P\{M_H = M_Z \text{ and } \mathbf{n}_k = \mathbf{m}_k \text{ for } k \leq M_H\} \geq \beta_\varepsilon \quad (3.2)$$

where $\beta_\varepsilon = 1 - 2\varepsilon^4 - e^{-\frac{1}{2\varepsilon^2}}$.

Proof. Let $A_k = \{\tau_k = \zeta_k \text{ and } \mathbf{n}_k = \mathbf{m}_k\}$ and $B_k = \cap_{l \leq k} A_l$. Then, as

$$\zeta_{k+1} \geq \zeta_k + T_{\partial V_{\mathbf{m}_k}^\varepsilon} \circ \theta_{\zeta_k},$$

by strong Markov property and (3.1), $P\{B_{k+1}|B_k\} \geq 1 - \varepsilon^{10}$. Therefore

$$P\{B_{[\varepsilon^{-6}]}\} \geq (1 - \varepsilon^{10})^{\varepsilon^6}.$$

Since $\varepsilon \in (0, \frac{1}{4})$ and $\log(1 - x) \geq -2x$ for $x \in (0, \frac{1}{2})$ we therefore have

$$\varepsilon^6 \log(1 - \varepsilon^{10}) \geq -2\varepsilon^4$$

so that

$$P\{B_{[\varepsilon^{-6}]}\} \geq e^{-2\varepsilon^4} \geq 1 - 2\varepsilon^4.$$

On the other hand

$$P\{M_H > \varepsilon^6\} \leq e^{-\frac{1}{2\varepsilon^2}}$$

so that

$$\begin{aligned} P\{M_H = M_Z \text{ and } \mathbf{n}_k = \mathbf{m}_k \text{ for } k \leq M_H\} \\ &\geq P\{B_{[\varepsilon^{-6}]}\} - P\{M_H > \varepsilon^6\} \\ &\geq 1 - 2\varepsilon^4 - e^{-\frac{1}{2\varepsilon^2}} \end{aligned}$$

which proves the lemma. ■

4 Proof of Theorem 1.2: using the signatures

This section is devoted to the proof of Theorem 1.2 by using information of its (extended) Stratonovich signatures. To this end, we need to choose a good version of multiple iterated Stratonovich's integrals.

Recall that $(\mathbf{W}, \mathcal{B}, P)$ is the classical Wiener space, where \mathbf{W} is the sample space of all continuous paths started at 0, on which the coordinate process $(W_t)_{t \geq 0}$ is Brownian motion under probability measure P . For each path $w \in \mathbf{W}$, and natural number n , we consider its dyadic approximations $w^{(n)} \in \mathbf{W}$ defined to be the polygon assuming the same values as w at dyadic points $\frac{j}{2^n}$ (for $j \in \mathbb{Z}_+$). According to Wong-Zakai [12] and Ikeda-Watanabe [8], there is a subset $\mathcal{N} \subset \mathbf{W}$ with probability zero, such that

$$\lim_{n \rightarrow \infty} \int_{s < t_1 < \dots < t_k < t} \alpha^1(dw_{t_1}^{(n)}) \cdots \alpha^k(dw_{t_k}^{(n)})$$

exists for every $w \in \mathbf{W} \setminus \mathcal{N}$, for all smooth differential forms α^j with bounded derivatives and for every pair $s < t$. The previous limit is denoted by

$[\alpha^1 \cdots \alpha^n](w)_{s,t}$. We fix such an exceptional set \mathcal{N} , and assign $[\alpha^1 \cdots \alpha^n](w)$ to be zero for $w \in \mathcal{N}$. The important fact is that $[\alpha^1 \cdots \alpha^n]_{s,t}$ is a version of Stratonovich's iterated integral

$$\int_{s < t_1 < \cdots < t_k < t} \alpha^1(\circ dW_{t_1}) \cdots \alpha^k(\circ dW_{t_k}).$$

In Lyons and Qian [9], a specific exceptional set \mathcal{N} was constructed by means of the so-called p -variation metric, which is however not needed in our proof of the main theorem.

In this section $[\alpha^1 \cdots \alpha^n]$ denotes the version of Stratonovich's iterated integral $[\alpha^1 \cdots \alpha^n]_{0,1}$ defined as above, so that $[\alpha^1 \cdots \alpha^n] = 0$ on \mathcal{N} .

Our goal is to show that W_t for all $t \leq 1$ is \mathcal{G}_1 -measurable. For $\varepsilon \in (0, 1/4)$, and choose α and β big enough so that the estimates in Lemmata 3.1 and 3.2 hold. Choose a smooth 1-form on \mathbb{R}^2 , $\phi(x_1, x_2) = f(x_1, x_2)dx_1$, with a compact support in Z_0^ε such that $f(x_1, x_2) = x_2$ on K_0^ε . For each $\mathbf{z} \in \mathbb{Z}^2$, let $\phi^{\mathbf{z}} = \phi(\cdot - \varepsilon \mathbf{z})$ (or $\phi^{\mathbf{z}, \varepsilon}$ if we wish to indicate the dependence on ε) be the translation of ϕ with compact support in $Z_{\mathbf{z}}^\varepsilon$. Therefore, $\{\phi^{\mathbf{z}} : \mathbf{z} \in \mathbb{Z}^2, \varepsilon \in (0, 1/4)\}$ is a countable family of non-trivial differential forms with disjoint compacts for every fixed ε . The key idea, as we have explained in the Introduction, is to read out the blocks $Z_{\mathbf{n}_l}^\varepsilon$'s which have been visited by the Brownian motion by using the extended Stratonovich's signatures of form $[\phi^{\mathbf{z}_1} \cdots \phi^{\mathbf{z}_m}]$.

Let $m \geq 0$. A finite ordered sequence (or called a word) of length $m+1$, $\langle \mathbf{z}_0 \cdots \mathbf{z}_m \rangle$ (where all \mathbf{z} 's belong to the lattice \mathbb{Z}^2), is admissible if $\mathbf{z}_l \neq \mathbf{z}_{l+1}$ for $l = 0, \dots, m-1$. Let \mathcal{W}_m denote the set of all admissible words of length $m+1$.

If $w \in \mathbf{W}$,

$$\hat{M}(w) = \sup \{m : [\phi^{\mathbf{z}_0} \cdots \phi^{\mathbf{z}_m}](w) \neq 0 \text{ for some } \langle \mathbf{z}_0 \cdots \mathbf{z}_m \rangle \in \mathcal{W}_m\}$$

so that \hat{M} is \mathcal{G}_1 -measurable. For each $m \in \mathbb{N}$ and each admissible word $\langle \mathbf{z}_0 \cdots \mathbf{z}_m \rangle \in \mathcal{W}_m$ define

$$\mathbf{A}_{m, \langle \mathbf{z}_0 \cdots \mathbf{z}_m \rangle} = \{\hat{M}(w) = m \text{ and } [\phi^{\mathbf{z}_0} \cdots \phi^{\mathbf{z}_m}](w) \neq 0\}. \quad (4.1)$$

Since $\phi^{\mathbf{z}}$ have disjoint supports, therefore, if $\zeta_{m+1}(w) > 1$, then $\hat{M}(w)$ can not be greater than m , so that $\hat{M} \leq M_Z$ except on the exceptional set \mathcal{N} . On the other hand, according to Lemma 2.3 and the strong Markov property, $\hat{M} \geq M_H$ almost surely. Therefore $M_H \leq \hat{M} \leq M_Z$ almost surely.

If $\hat{M}(w) = m$, there is at most one $\langle z_0 \cdots z_m \rangle \in \mathcal{W}_m$ such that $[\phi^{z_0} \cdots \phi^{z_m}](w) \neq 0$ and all other $[\phi^{z'_0} \cdots \phi^{z'_n}](w) = 0$ for $\langle z'_0 \cdots z'_n \rangle \in \mathcal{W}_n$ if $n > m$ or if $n = m$ but $\langle z'_0 \cdots z'_m \rangle \neq \langle z_0 \cdots z_m \rangle$.

Let

$$\tilde{\mathbf{W}}_{m, \langle z_0 \cdots z_m \rangle} = \{M_H = m, \mathbf{n}_l = z_l \text{ for } l = 0, \dots, m\}. \quad (4.2)$$

for each admissible word $\langle z_0 \cdots z_m \rangle \in \mathcal{W}_m$, and

$$\tilde{\mathbf{W}}_\varepsilon = \bigcap_{m=0}^{\infty} \bigcup_{\langle z_0 \cdots z_m \rangle \in \mathcal{W}_m} \tilde{\mathbf{W}}_{m, \langle z_0 \cdots z_m \rangle}.$$

Then, according to Lemma 3.2, $P(\tilde{\mathbf{W}}_\varepsilon) \geq \beta_\varepsilon$.

We are now in a position to complete our proof. Set

$$\tilde{\mathbf{n}}_l = \sum_{m=0}^{\infty} \sum_{\langle z_0 \cdots z_m \rangle \in \mathcal{W}_m} z_l 1_{A_{m, \langle z_0 \cdots z_m \rangle}}$$

and redefine

$$\hat{w}(\varepsilon)_t = \tilde{\mathbf{n}}_l \varepsilon + \frac{t - \tau_l}{\tau_{l+1} - \tau_l} \tilde{\mathbf{n}}_{l+1} \varepsilon \quad \text{if } t \in [\tau_l, \tau_{l+1}]$$

then, we may choose a sequence $\varepsilon_n \downarrow 0$ so that $\sum_n (1 - \beta_{\varepsilon_n}) < \infty$. Then, $\hat{w}(\varepsilon_n) = w(\varepsilon_n)$ almost surely on $\tilde{\mathbf{W}}_\varepsilon$. Since $P(\tilde{\mathbf{W}}_\varepsilon) \geq \beta_\varepsilon$, it follows the Borel-Cantelli lemma, $\sup_{t \in [0,1]} |\hat{w}(\varepsilon_n) - w(\varepsilon_n)| \rightarrow 0$ in probability as $n \rightarrow \infty$, and therefore $W_t \in \mathcal{G}_1$ for $t \leq 1$.

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